Math 116, Homework 6

Q1. Evaluate the iterated integral: \( \int_0^1 \int_0^{1-x} \int_0^{2-x} xyz \, dz \, dy \, dx \).

Q2. Show that the volume of the sphere \( x^2 + y^2 + z^2 = a^2 \) in:
   a) cylindrical coordinates can be evaluated by the triple integral;
   \[ V = 2 \int_0^{2\pi} \int_0^a \int_{\sqrt{a^2-r^2}}^0 rdz \, dr \, d\theta. \]
   b) spherical coordinates can be evaluated by the triple integral;
   \[ V = \int_0^{2\pi} \int_0^\pi \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta. \]

Q3. Given the vector field \( \vec{F} = xy \vec{i} + xy^2 \vec{j} \).
   Is it conservative? If yes, find the potential function.

Q4. Show that, if the vector field \( \vec{F}(x, y) = F_1(x, y) \vec{i} + F_2(x, y) \vec{j} \) is conservative in a domain \( D \), then the condition \( \frac{\partial F_1(x, y)}{\partial y} = \frac{\partial F_2(x, y)}{\partial x} \) is satisfied at all points of \( D \).

Q5. Show that the vector field \( \vec{F}(x, y) = x \vec{i} - y \vec{j} \) is conservative and find the potential function for it.

Q6. Show that the vector field \( \vec{F}(x, y, z) = 2xz \vec{i} + 2yz \vec{j} - x^2 + y^2 z^2 \vec{k} \) is conservative and find the potential function for it.

Q7. Determine whether or not the following fields are conservative. If the field is conservative, find its potential function. (Note: there are two tests for conservative fields. Make sure you know them both);
   a) \( \vec{F}(x, y, z) = (x + y) \vec{i} + z \vec{j} + (x + y) \vec{k} \)
   b) \( \vec{F}(x, y, z) = y \sin z \vec{i} + x \sin z \vec{j} + xy \cos z \vec{k} \)

Q8. Consider the vector field \( \vec{F}(x, y, z) = e^y \vec{i} + (xe^y + z^2) \vec{j} + (2yz - 1) \vec{k} \).
   a) Compute \( \text{Curl} \vec{F} \)
   b) Use part (a), show that \( \vec{F} \) is conservative.

Q9. Let \( C \) be the curve along the graph of \( y = x^3 - x^2 \) between the points \((0, 0)\) and \((1, 0)\).
   a) Give a parametrization for \( C \),
   b) \( \int_C \vec{F} \cdot d\vec{r} \)
Q10. Consider the vector field:
\[ F(x, y, z) = (Ax \sin(\pi y))\vec{i} + (x^2 \cos(\pi y) + Bye^{-z})\vec{j} + (y^2 e^{-z})\vec{k}. \]
a) For which values of the constants \( A \) and \( B \) is the vector field conservative?
Compute \( \text{Curl} \vec{F} \)
b) Find \( \int_C \vec{F} \cdot d\vec{r} \), where \( C \) is the curve \( \vec{r} = (\cos t)\vec{i} + (\sin 2t)\vec{j} + (\sin^2 t)\vec{k} \).

Q11. Evaluate
\[
\int_C (e^{x+y} \sin(y + z))dx + e^{x+y}(\sin(y + z) + \cos(y + z))dy + (e^{x+y} \cos(y + z))dz
\]
along the straight line segment from \((0, 0, 0)\) to \((1, \frac{\pi}{4}, \frac{\pi}{4})\).

Q12. Find the work done by the force field:
\[ F(x, y, z) = (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + (3xz^2 + 2)\vec{k} \]
in moving a particle along the curve \( x = \arcsin t, y = 1 - 2t, z = 3t - 1, (0 \leq t \leq 1) \).

Q13. Evaluate the double integral \( \int_R \int x dA \)
by converting to polar coordinates, where \( R \) is the region in the first quadrant bounded by the lines \( x = 0, y = x, x^2 + y^2 = 9 \).