Math 115, Homework 5

Q1. Evaluate each of the following limits;
   a) \( \lim_{x \to 0} \frac{x - \tan^2 x}{\sin x} \)
   b) \( \lim_{x \to \frac{\pi}{4}} \frac{\pi(x - \frac{\pi}{4})}{\sin(\pi - 4x)} \)
   c) \( \lim_{x \to 2} \frac{\sqrt{4x + 1} - \sqrt{x + 7}}{x - 2} \)

Q2. Evaluate each of the following limits;
   a) \( \lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{\cos 2x} \)
   b) \( \lim_{x \to \infty} (\sqrt{x^2 + x} - \sqrt{x^2 - 4}) \)
   c) \( \lim_{x \to -1} \frac{2x^3 - 2x^2 - x + 1}{x^3 - 4x^2 + 10x - 7} \)

Q3. Evaluate each of the following limits;
   a) \( \lim_{x \to 3} \frac{\sqrt{x + 6} - 3}{\sqrt{2x + 10} - 4} \)
   b) \( \lim_{x \to \infty} (\sqrt{x^2 + 3x - 2} - x) \)

Q4. Find \( f'(1) \), (if exists) for the function;
   \[ f(x) = \begin{cases} 
   x^2, & x \geq 1 \\
   2x - 1, & x < 1.
   \end{cases} \]

Q5. Find derivatives of the following functions at indicated points;
   a) \( f(x) = [x - 1], x = 2 \) and \( x = \frac{5}{2} \).
   b) \( f(x) = |x^2 - 1|, x = -1 \) and \( x = \sqrt{2} \)

Q6. By using \( f'(x) \), write an equivalent form of the following limits;
   a) \( \lim_{h \to 0} \frac{f(x - h) - f(x)}{h} \)
   b) \( \lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} \)
   c) \( \lim_{h \to 0} \frac{f(x + 3h) - f(x)}{h} \)

Q7. Prove that, if \( f(x) \) and \( g(x) \) are differentiable functions at \( x_0 \), then
   \[ (f.g)'(x_0) = f'(x_0).g(x_0) + f(x_0).g'(x_0) \]

Q8. Let
   \[ f(x) = \begin{cases} 
   \cos x, & -\frac{\pi}{2} \leq x < \frac{\pi}{4} \\
   2 - x, & \frac{\pi}{4} \leq x < 3 \\
   x - 2, & 3 \leq x \leq 4
   \end{cases} \]
   a) Is \( f(x) \) continuous at \( x = \frac{\pi}{4} \)?
   b) Is \( f(x) \) differentiable at \( x = \frac{\pi}{4} \) and \( x = 3 \)?
Q9. Let

\[ f(x) = \begin{cases} 
  [2x - 1], & x \geq 2 \\
  \frac{4-x}{x^2}, & 2 \geq x > 3 \\
  \sin \pi x, & 0 > x
\end{cases} \]

a) Is \( f(x) \) continuous at \( x = 0, x = 1 \) and \( x = 2 \)?
b) Is \( f(x) \) differentiable at \( x = 0, x = 1 \) and \( x = 2 \)?

Q10. Assume that \( f(x) \) is a differentiable function satisfying the equation,

\[ f(x + y) = f(x) + f(y) + 2xy \]

for all real numbers \( x \) and \( y \). Suppose also that

\[ \lim_{x \to 0} \frac{f(x)}{x} = 2. \]

Find \( f'(0) \).

Q11. Evaluate each of the following limits;

a) \( \lim_{x \to 0} \frac{|3x + 1| - |3x - 1|}{5x} \)

b) \( \lim_{x \to \frac{\pi}{4}} \frac{\sin (x + \frac{\pi}{4}) - 1}{x - \frac{\pi}{4}} \)